Qualifying Exam   Complex Analysis   May, 2018

1. Prove that there exists a unique entire function f that satisfies $f(z) - zf''(z) = 1$. Hint: Consider the power series expansion of f. Note that you need to show that the series does converge at every point in the plane.

2. Let f and g be two entire functions. Suppose that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Prove that there is a constant $C \in \mathbb{C}$ with $|C| \leq 1$ such that $f(z) = Cg(z)$ for all $z \in \mathbb{C}$. Hint: Consider the function f/g.

3. Evaluate the integral
$$\int_0^\infty \frac{\log x}{x^4+1}\, dx.$$

4. Let f be an injective analytic function defined on $\mathbb{C} \setminus \{0\}$. Prove that there is some $w_0 \in \mathbb{C}$ such that $f(\mathbb{C} \setminus \{0\}) = \mathbb{C} \setminus \{w_0\}$.

5. Let (f_n) be a sequence of analytic functions on a domain U, which is uniformly bounded. Assume that for each $z \in U$, the sequence $(f_n(z))$ converges. Show that there is a function f analytic on U such that (f_n) converges to f uniformly on every compact subset of U.